Dynamics versus replicas in the random field Ising model

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Abstract

In a previous article we have shown, within the replica formalism, that the conventional picture of the random field Ising model breaks down, by the effect of singularities in the interactions between fields involving several replicas, below dimension eight. In the zero-replica limit several coupling constants have thus to be considered, instead of just one. As a result we had found that there is no stable fixed point in the vicinity of dimension six. It is natural to reconsider the problem in a dynamical framework, which does not require replicas, although the equilibrium properties should be recovered in the large time limit. Singularities in the zero-replica limit are a priori not visible in a dynamical picture. In this note we show that in fact new interactions are also generated in the stochastic approach. Similarly these interactions are found to be singular below dimension eight. These critical singularities require the introduction of a time origin t_0 at which initial data are given. The dynamical properties are thus dependent upon the waiting time. It is shown here that one can indeed find a complete correspondence between the equilibrium singularities in the $n = 0$ limit, and the singularities in the dynamics when the initial time t_0 goes to minus infinity, with n replaced by $-\frac{1}{t_0}$. There is thus complete coherence between the two approaches.

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1   Introduction

In a recent article [1] we have reconsidered the old problem of the random field Ising model (RFIM). Using the replica approach, we started with a ϕ^4 field theory, but included the five marginal operators of same dimension, namely $g_1 \sum \phi_a^4, g_2 \sum \phi_a^3 \sum \phi_b, g_3 (\sum \phi_a^2)^2, g_4 \sum \phi_a^2 (\sum \phi_b)^2, g_5 (\sum \phi_a)^4$, in which the sums run from 1 to n. The last four operators are normally not considered, since they are a priori of relative order, n, n^2 or n^3 with respect to the first operator. Within this naive $n = 0$ limit one recovers the usual correspondence, between the RFIM in dimension d and the pure system in dimension $(d - 2)$ [2, 3, 4], which is known to break down in three dimensions [5, 6].

However we have shown in [1] that the statistical fluctuations give singular contributions to the effective interactions between several replicas g_i^{eff}, with $2 \le i \le 5$. They are singular below dimension eight in the critical region when n goes to zero, invalidating thus the naive zero-replica limit. For instance g_3^{eff} receives a one-loop contribution due to g_1 which is proportional (at zero momentum) to

$$I_n(r) = g_1^2 \int d^d q \; \frac{1}{(q^2 + r)^2} \; \frac{\Delta^2}{(q^2 + r - n\Delta)^2} \tag{1.1}$$

in which Δ is the variance of the Gaussian random field, and r is proportional to the distance to the critical temperature. In the critical region in which r, which goes to zero, is of order $n\Delta$, $I_n(n\Delta)$ diverges for n small in dimensions lower than eight as

$$I_n(n\Delta) \sim \frac{1}{n^{(8-d)/2}}. \tag{1.2}$$

As a result g_3^{eff} is of order $1/n$ in dimension six and this invalidates the simple $n = 0$ limit. Consequently we have studied the renormalization group flow in dimension $6 - \epsilon$ and found that the usual dimensional reduction fixed point is unstable, as well as all the other fixed points allowed by this flow [1].

In this work we want to reconsider the same problem, within the dynamical approach which is free of replicas, and is then physically more transparent. Formally, J. Kurchan [7] has shown, within the supersymmetric approach [12], that one gets exactly the same action with fields bearing replica indices (for the statics) and with fields bearing time-indices (in the dynamics), except

for the time derivative term. However the detailed relationship between the two formalisms is far from clear, especially whenever singularities associated with the number of replicas are involved. Since equilibrium is the large time limit of a stochastic process, the above mechanism ought to be understood as well in a dynamical framework [8]. We have thus considered a Langevin stochastic equation, and the corresponding field theory obtained through the averaging over the noise and over the random field. In this approach we find also singularities below dimension eight related to new effective dynamic interactions. In order to cope with these singularities, we have to introduce a "waiting" time, i.e. we assume that we start with some initial data at finite time t_0. The singularities manifest themselves then as divergences when t_0 goes to minus infinity, with a correspondence between $-t_0$ and $1/n$. We believe that this supports significantly the results of the analysis that we had developped within the replica approach [1].

2   The Langevin formalism

We consider a simple Langevin dynamics for a ϕ^4 theory,

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{\delta S}{\delta \phi} + h(x) + \eta(x,t), \tag{2.1}$$

with a white noise source

$$< \eta(x,t)\eta(x',t') > = \gamma\delta(t-t')\delta^{(d)}(x-x'), \tag{2.2}$$

a time-independent random field,

$$\overline{h(x)} = 0 \ , \ \overline{h(x)h(y)} = \Delta\delta^{(d)}(x-y). \tag{2.3}$$

for an action

$$S(\phi) = \int dx[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + \frac{g}{4!}\phi^4]. \tag{2.4}$$

From this, one derives easily the bare (i.e. $g = 0$) correlator defined as

$$C(t,t',p;t_0) = \int d^dx \exp{(ip \cdot x)} \, \overline{< \phi(x,t)\phi(0,t') >} \tag{2.5}$$

with initial data

$$\phi(x,t_0) = 0. \tag{2.6}$$

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The result is

$$C(t, t', p; t_0) = \frac{\gamma}{2E_p}[e^{-E_p|t-t'|} - e^{-E_p(t+t'-2t_0)}] + \frac{\Delta}{E_p^2}[1 - e^{-E_p(t-t_0)}][1 - e^{-E_p(t'-t_0)}]$$

(2.7)

with

$$E_p = p^2 + r_0 .$$

(2.8)

If we let the initial time t_0 go to minus infinity, we recover the time-translation invariant correlator

$$C(t, t', p; -\infty) = \frac{\gamma}{2E_p}e^{-E_p|t-t'|} + \frac{\Delta}{E_p^2} ,$$

(2.9)

and its Fourier transform with respect to $(t - t')$ is

$$\tilde{C}(\omega, p) = \frac{\gamma}{2\pi}\frac{1}{(p^2 + r_0)^2 + \omega^2} + \frac{\Delta}{(p^2 + r_0)^2}\delta(\omega).$$

(2.10)

The field theory related to this stochastic process is derived in the standard way [10, 12]:

$$
\begin{aligned}
< O(\phi) > &= \int D\phi O(\phi) \det(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta \phi^2}) \overline{< \prod_{x,t} \delta(\frac{\partial \phi(x, t)}{\partial t} + \frac{\delta S}{\delta \phi} - h(x) - \eta(x, t)) >} \\
&= \int D\phi D\hat{\phi} Dc D\bar{c} O(\phi) \exp - \int dx dt \bar{c}\left(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta \phi^2}\right) c \\
&\quad \overline{< \exp i \int dx dt \hat{\phi}(\frac{\partial \phi(x, t)}{\partial t} + \frac{\delta S}{\delta \phi} - h(x) - \eta(x, t)) >} \\
&= \int D\phi D\hat{\phi} Dc D\bar{c}\ O(\phi)\ \exp - \int dx dt \bar{c}\left(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta \phi^2}\right) c \\
&\quad \exp i \int dx dt \hat{\phi}(\frac{\partial \phi(x, t)}{\partial t} + \frac{\delta S}{\delta \phi}) \\
&\quad \exp - \frac{\gamma}{2}\int dx dt \phi(\hat{x}, t)^2 - \frac{\Delta}{2}\int dx \int dt \int dt' \phi(\hat{x}, t)\phi(\hat{x}, t').
\end{aligned}
$$

(2.11)

This is summarized in an effective action

$$
\begin{aligned}
S_{eff} &= \int dx dt\ \bar{c}\left(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta \phi^2}\right) c - i\int dx dt \hat{\phi}(x, t)\left(\frac{\partial \phi(x, t)}{\partial t} + \frac{\delta S}{\delta \phi}\right) \\
&\quad + \frac{\gamma}{2}\int dx dt \hat{\phi}^2(x, t) + \frac{\Delta}{2}\int dx \int dt \int dt' \hat{\phi}(x, t)\hat{\phi}(x, t'),
\end{aligned}
$$

(2.12)

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in which $c(x, t)$ and $\bar{c}(x, t)$ are Grassmanian fields.

The quadratic part of this effective action allows one to recover the $\phi\phi$ correlator (2.7), as well as the response

$$R(p, t, t') = \int d^d x < \phi(x, t)\hat{\phi}(0, t') >= i\theta(t - t') \exp -(E_p(t - t')). \quad (2.13)$$

Note that the effective action (2.12) contains only one interaction term, local in time, proportional to $\hat{\phi}(x, t)\phi^3(x, t)$.

3 Singularities below dimension eight

Let us assume first that we let the initial time t_0 go to minus infinity, and work with the simple correlator (2.7). We consider the lowest order contribution to the four-point function $\phi\hat{\phi}\phi\hat{\phi}$ at zero-momentum, zero frequency. It consists of a simple one-loop graph, which involves the integral of the product of two correlators (2.9). The random-field part of the correlator (2.9) gives a contribution proportional to $g^2\Delta^2 \int d^d p \frac{1}{(p^2+r)^4}$ which diverges in the critical domain for $d < 8$. (Note that in the static replica approach one has an identical critical singularity for the zero-momentum $\phi_a^2\phi_b^2$ function.)

In order to deal properly with this singularity we keep a finite initial time t_0 and compute again the lowest order contribution to the four-point function $\phi(t)\hat{\phi}(t)\phi(t')\hat{\phi}(t')$ at zero-momentum. It consists of an integral over the momenta only of the product of two correlators (2.7). Keeping only the Δ^2 part we have to consider

$$I(t, t'; t_0) = g^2\Delta^2 \int d^d p \frac{1}{E_p^4}[1 - e^{-E_p(t-t_0)}]^2[1 - e^{-E_p(t'-t_0)}]^2, \quad (3.1)$$

with $E_p = p^2 + r$. If we let t_0 go to $-\infty$, we recover the previous critical singularity in dimensions lower than eight as it should. If we keep t_0 finite, but let it grow to large negative values, we have to integrate out (3.1) over p. Using the identity

$$\frac{1}{E}[1 - e^{-E(t-t_0)}] = \int_0^{t-t_0} d\tau e^{-E\tau}, \quad (3.2)$$

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we end up after a few steps with

$$I(t, t'; t_0) \sim (-t_o)^{(8-d)/2} f(u), \qquad (3.3)$$

in which $f(u)$ is the function of the single variable

$$u = (-t_0)r \qquad (3.4)$$

given by

$$f(u) = u^{\frac{d-8}{2}} \int_0^\infty dx \frac{x^{\frac{d}{2}-1}}{(1+x)^4} [1 - e^{u(1+x)}]^4. \qquad (3.5)$$

Therefore for finite, but large, negative t_0, when one enters in the critical region with r of order $1/|t_0|$ the four-point function $\phi(t)\hat{\phi}(t)\phi(t')\hat{\phi}(t')$ receives contributions of order $|t_o|^{(8-d)/2}$ independent of t and t'. A new interaction vertex, proportional to $|t_o|^{(8-d)/2} \int dx \int dt dt' \phi(x,t)\hat{\phi}(x,t)\phi(x,t')\hat{\phi}(x,t')$, not present in the initial Langevin dynamics is thus generated, in very much the same way that in the replica approach a vertex of the form $n^{(8-d)/2} \int dx \left[\sum_a \phi_a^2(x)\right]^2$ resulted from the singularity in less than eight dimensions.

A similar analysis of the dynamical fluctuations at two-loop order lead to three more singular contributions with four external legs. They are respectively proportional to

$| t_o |^{(8-d)/2} \int dx \int dt dt' dt'' \phi(x,t)\hat{\phi}(x,t)\phi(x,t')\phi(x,t'')\theta(t-t')$,

$| t_o |^{(8-d)} \int dx \int dt dt' dt'' \phi(x,t)\hat{\phi}(x,t)\hat{\phi}(x,t')\hat{\phi}(x,t'')$,

$| t_o |^{\frac{3(8-d)}{2}} \int dx \left(\int dt \hat{\phi}(x,t) \right)^4$.

We have thus a total of five dynamical interactions of degree four in the operators ϕ and $\hat{\phi}$. This is similar to the five operators that we had to consider in the replica approach, namely $\sum_a \phi_a^4$, $n^{-\frac{8-d}{2}}(\sum_a \phi_a^2)^2$, $n^{-\frac{8-d}{2}}(\sum_a \phi_a^3)\sum_b \phi_b$, $n^{-(8-d)}(\sum_a \phi_a^2)(\sum_b \phi_b)^2$, $n^{-\frac{3(8-d)}{2}}(\sum_a \phi_a)^4$. Clearly $1/n$ is replaced in the dynamics by $|t_0|$.

It is interesting to analyse the renormalization group flow with those five operators. Some diagrams, which would vanish in the replica approach because they have an explicit relative factor of n, in spite of the singular behavior of the interactions, are identically zero in the dynamics by causality; this is due to the retarded nature of the response functions which forbids to make closed loops by propagating only response functions. Whenever the

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singular behavior leads to a finite result in the $n = 0$ limit of the replica approach, the singular dependence in t_0 does lead to the same conclusion in the dynamics.

4   Conclusion

We have shown that the dynamical approach to the dynamics of a random field Ising model (or ϕ^4 theory), shows that new singular interactions, absent from the bare dynamical action corresponding to the Langevin equation, are generated by the fluctuations below dimension eight.These new interactions cannot be easily cast into a modified Langevin equation. These interactions have singularities in the waiting time, analogous to the singularities that we had found earlier in the replica approach, with a simple correspondence between $1/n$ and $|t_0|$. The conclusions that we had drawn from our earlier work, namely that in the space of the five coupling constants there was no stable fixed point of order $\epsilon = 6 - d$, are thus entirely supported by the dynamics.

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